

Geodesic flows in rotating black hole backgrounds

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Abstract

We study the kinematics of timelike geodesic congruences, in the spacetime geometry of rotating black holes in three (the BTZ) and four (the Kerr) dimensions. The evolution (Raychaudhuri) equations for the expansion, shear and rotation along geodesic flows in such spacetimes are obtained. For the BTZ case, the equations are solved analytically. The effect of the negative cosmological constant on the evolution of the expansion (θ), for congruences with and without an initial rotation (ω_0) is noted. Subsequently, the evolution equations, in the case of a Kerr black hole in four dimensions are written and solved numerically, for some specific geodesics flows. It turns out that, for the Kerr black hole, there exists a critical value of the initial expansion below (above) which we have focusing (defocusing). We delineate the dependencies of the expansion, on the black hole angular momentum parameter, a , as well as on ω_0 . Further, the role of a and ω_0 on the time (affine parameter) of approach to a singularity (defocusing/focusing) is studied. While the role of ω_0 on this time of approach is as expected, the effect of a leads to an interesting new result.

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I. INTRODUCTION

Geodesic congruences (flows) and their features in various spacetime backgrounds have been extensively studied by mathematicians and physicists alike, over many years. In the context of gravity and general relativity, studies on the evolution (along a chosen geodesic flow) of the kinematical quantities (the isotropic expansion, shear and rotation (henceforth referred as ESR)) in specific spacetime geometries, have been carried out. The ESR evolution, as is well-known, is governed by the Raychaudhuri equations [1, 2]. In particular, consequences (geodesic focusing) of the Raychaudhuri equation for the expansion scalar are crucial while proving the singularity theorems in general relativity [3, 4].

Recently, we have initiated a programme where we intend to study geodesic flows in various contexts (an example outside the realm of general relativity, is, deformable media) by solving the evolution equations for the ESR, in various scenarios. Some of our work is available in [5]-[8].

In order to push things further beyond our investigations in [7], here we intend to look at geodesic flows in rotating black hole spacetimes. To start with, we restrict to three dimensions, where the well-known black hole solution (with rotation) is the Bañados–Teitelboim–Zanelli (BTZ) line element [10, 11]. The BTZ, is a solution of the three dimensional Einstein equations with a negative cosmological constant. Further, in four dimensions, we have the famous Kerr black hole [1, 2] which is also important, astrophysically. Such black holes are characterised by their mass and spin (rotation rate) and the study of geodesic flows in their gravitational fields can provide useful information about the behaviour of matter around strong gravitational fields.

In this article, we aim towards understanding the kinematics of geodesic flows in the above mentioned spacetimes, in three and four dimensions. Our article is organised as follows. We first review the BTZ spacetime along with its geodesic structure in Section II. The Raychaudhuri equations in the background of the BTZ black hole spacetime are then written in a freely falling Fermi normal frame. Surprisingly, the Raychaudhuri equations in this case are analogous to those in the flat background, with stiffness [5] identified with the modulus of the cosmological constant. The analytical solutions are discussed and the kinematics of flows is revealed accordingly in Section III. Next, the Kerr black hole spacetime geometry (which is a vacuum, axisymmetric solution of the Einstein equations) with

the associated geodesic equations is reviewed briefly in Section IV followed by the derivation of Raychaudhuri equations in Section V. In view of the absence of the analytical solutions of the (geodesic and Raychaudhuri) equations for the Kerr case, we have solved them numerically. Based on the evolution of ESR variables for various sets of initial conditions with different values of the other parameters involved, the generic features of the geodesic flows are visualised in Section V. The results are also compared with those obtained for the Schwarzschild geometry. Finally, in Section VI, we highlight the important results obtained and indicate the possibilities for future work.

II. BTZ BLACK HOLE SPACETIME

Let us begin by writing down the line element for the BTZ black hole in (2+1) dimensions [10, 11],

$$ds^2 = -(N^2 - r^2 N_\phi^2) dt^2 + N^{-2} dr^2 + r^2 d\phi^2 + 2r^2 N_\phi dt d\phi, \quad (2.1)$$

Here, the squared lapse function N^2 and the angular shift N_ϕ are given by,

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}; \quad N_\phi = -\frac{J}{2r^2}, \quad (2.2)$$

with $-\infty < t < \infty$, $0 < r < \infty$, and $0 \leq \phi \leq 2\pi$. In equation (2.2), M and J are, respectively, the mass and angular momentum parameters of the BTZ black hole, while the parameter l is the radius of curvature of the spacetime geometry, related to the cosmological constant (Λ) as $l^2 = -1/\Lambda$. The lapse function vanishes for the following two values of r ,

$$r_{\pm} = l \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}} \right) \right]^{\frac{1}{2}}, \quad (2.3)$$

where r_{\pm} identifies the horizons of the BTZ black hole, which exist only when the following conditions are satisfied,

$$M > 0; \quad |J| \leq Ml. \quad (2.4)$$

The inner (r_-) and outer (r_+) horizons shrink with the increasing value of the cosmological constant. One may also note here that both the roots of the lapse function coincide for the extremal case $|J| = Ml$. The first integral of the geodesic equations corresponding to the line element (2.1) are given as follows [12, 13],

$$\dot{t} = -\frac{1}{N^2} (E + N_\phi L), \quad (2.5)$$

$$\dot{\phi} = \frac{N_\phi}{N^2} (E + N_\phi L) - \frac{L}{r^2}, \quad (2.6)$$

$$\dot{r} = \pm \left[(E + N_\phi L)^2 - \frac{N^2 L^2}{r^2} - N^2 \right]^{\frac{1}{2}}, \quad (2.7)$$

where E and L are integration constants which represent the energy and angular momentum, respectively. Here $u^i = (\dot{t}, \dot{r}, \dot{\phi})$ satisfies the timelike constraint $u^i u_i = -1$ for the timelike geodesics. The equations (2.5)-(2.7) characterise the motion of the test particles in the BTZ black hole spacetime.

III. EVOLUTION OF ESR IN THE LOCAL FRAME

A. Fermi normal frame and the Raychaudhuri equations

We construct an orthonormal Fermi frame with u^i, \hat{e}_α^i ($\alpha = 1, 2$) where

$$g_{ij} \hat{e}_\alpha^i \hat{e}_\beta^j = \delta_{\alpha\beta} \quad ; \quad g_{ij} u^i \hat{e}_\alpha^j = 0 \quad (3.1)$$

This is the local inertial frame of a freely falling observer. The basis set $\{u^i, \hat{e}_\alpha^i\}$ is parallelly transported along the geodesic.

The tensor $B_{ij} = \nabla_j u_i$, defined in the coordinate frame, may be rewritten in the local frame of the freely falling observer as $B_{\alpha\beta} = B_{ij} \hat{e}_\alpha^i \hat{e}_\beta^j$. Further, $B_{\alpha\beta}$ can be decomposed into its trace, symmetric traceless and antisymmetric parts in the usual way,

$$B_{\alpha\beta} = \frac{1}{2} \theta \delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \quad (3.2)$$

where,

$$\frac{1}{2} \theta \delta_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} \theta & 0 \\ 0 & \frac{1}{2} \theta \end{pmatrix}, \quad \sigma_{\alpha\beta} = \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix}, \quad \omega_{\alpha\beta} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}. \quad (3.3)$$

In the above, θ is the expansion scalar, σ_+ and σ_\times are the components of trace-free shear tensor $\sigma_{\alpha\beta}$ while ω denotes the component of rotation in the antisymmetric tensor $\omega_{\alpha\beta}$. The evolution equation for $B_{\alpha\beta}$ in the Fermi normal frame as discussed above is then given as,

$$\dot{B}_{\alpha\beta} + B_{\alpha\gamma} B^\gamma_\beta = -R_{ikjl} u^k u^l \hat{e}_\alpha^i \hat{e}_\beta^j. \quad (3.4)$$

Since we are working in three dimensions, we can make use of the following relation between the Riemann, Ricci tensors and the Ricci scalar,

$$R_{ikjl} = g_{ij} R_{kl} + g_{kl} R_{ij} - g_{kj} R_{il} - g_{il} R_{jk} - \frac{1}{2} (g_{ij} g_{kl} - g_{il} g_{jk}) R \quad (3.5)$$

and also the relation $R_{ij} = \Lambda g_{ij}$. Consequently, the equation for $B_{\alpha\beta}$ becomes

$$\dot{B}_{\alpha\beta} + B_{\alpha\gamma} B_{\beta}^{\gamma} = \Lambda \delta_{\alpha\beta}. \quad (3.6)$$

Using (3.2), and (3.6), the Raychaudhuri equations for the ESR variables turn out to be,

$$\dot{\theta} + \frac{1}{2}\theta^2 + 2(\sigma_+^2 + \sigma_{\times}^2 - \omega^2) + 2|\Lambda| = 0, \quad (3.7)$$

$$\dot{\sigma}_+ + \theta\sigma_+ = 0, \quad (3.8)$$

$$\dot{\sigma}_{\times} + \theta\sigma_{\times} = 0, \quad (3.9)$$

$$\dot{\omega} + \theta\omega = 0. \quad (3.10)$$

The above first-order, coupled, nonlinear and inhomogeneous ordinary differential equations have to be solved, for given initial conditions, in order to obtain the kinematical evolution of the ESR along geodesic congruences.

A subtle point may be noted here. Given the (static) velocity vector field $u^i = (\dot{t}, \dot{\phi}, \dot{r})$ in (2.5)-(2.7) one can easily find the ESR variables as functions of r using the definition $B_{ij} = \nabla_j u_i$. This leads to, for example,

$$\theta = \nabla_i u^i = \pm \frac{l^2(E^2 + M) - L^2 - 2r^2}{l^2 r \left[-\frac{4L^2}{l^2} + (E^2 + M) + \frac{4L^2 M - J(4LE + J)}{r^2} - \frac{4r^2}{l^2} \right]^{1/2}}, \quad (3.11)$$

and the vorticity $\omega \equiv 0$. Physically, this represents the kinematics of a geodesic congruence in which all geodesics have a fixed value of the constants E and L . Therefore, this expression cannot accommodate arbitrary initial conditions on the velocity field of a congruence, or on the ESR variables. The solutions obtained by integrating the Raychaudhuri equations (3.7)-(3.10) directly, for given arbitrary initial conditions, are, in this sense more general, and thereby, underscores the importance of these equations.

B. Analytical solutions

The equations (3.7)-(3.10) have been exactly solved analytically (see [5, 6] for the detailed procedure). It is interesting to note from equations (3.7)-(3.10) here and (2.8)-(2.11) in [5] that, kinematically, the equations in the BTZ black hole background can be mapped to those for flows in a flat space with isotropic stiffness $|\Lambda|$ (k in [5]). In order to solve this set of equations, we first define a parameter $I = \sigma_+^2 + \sigma_{\times}^2 - \omega^2$ and obtain the following analytical

expressions for the ESR for the three different cases depending on the signature of I .

Case A: $I > 0$

$$\theta = \frac{ap \cos(c_1 + \frac{pt}{2})}{2[a \sin(c_1 + \frac{pt}{2}) + b]}, \quad (3.12)$$

$$\{\sigma_+, \sigma_-, \omega\} = \frac{\{c_2, c_3, c_4\}}{[a \sin(c_1 + \frac{pt}{2}) + b]}, \quad (3.13)$$

where $p^2 = 16|\Lambda|$ and $a^2 = 16 + b^2$. Here we have used $\{ \}$ brackets to express the solutions in a compact way for the components of shear and rotation which have a common denominator as in equation (3.13). The parameter b and the integration constants (c_1, c_2, c_3 and c_4) involved can be given in terms of the initial values of ESR (i.e., $\theta_0, \sigma_{+0}, \sigma_{-0}$ and ω_0) as described below,

$$b = \sqrt{\frac{I_0}{p^2}} \left(-8 + \frac{2\theta_0^2}{I_0} + \frac{p^2}{2I_0} \right), \quad (3.14)$$

$$c_1 = \tan^{-1} \left(\frac{p - b\sqrt{I_0}}{2\theta_0} \right), \quad (3.15)$$

$$\{c_2, c_3, c_4\} = \{\sigma_{+0}, \sigma_{-0}, \omega_0\} (a \sin c_1 + b). \quad (3.16)$$

Case B: $I = 0$

$$\theta = 2\sqrt{|\Lambda|} \tan[\sqrt{|\Lambda|}(c_1 - t)], \quad (3.17)$$

$$\{\sigma_+, \sigma_-, \omega\} = \{c_2, c_3, c_4\} \sec^2[\sqrt{|\Lambda|}(c_1 - t)]. \quad (3.18)$$

The integration constants in terms of the initial ESR for this case are as given below,

$$c_1 = \frac{1}{\sqrt{|\Lambda|}} \tan^{-1} \left(\frac{\theta_0}{2\sqrt{|\Lambda|}} \right), \quad (3.19)$$

$$\{c_2, c_3, c_4\} = \frac{\{\sigma_{+0}, \sigma_{-0}, \omega_0\}}{\sec^2(\sqrt{|\Lambda|} c_1)}. \quad (3.20)$$

Case C: $I < 0$

$$\theta = \frac{ap \sin(c_1 + \frac{pt}{2})}{2[a \cos(c_1 + \frac{pt}{2}) + b]}, \quad (3.21)$$

$$\{\sigma_+, \sigma_-, \omega\} = \frac{\{c_2, c_3, c_4\}}{[a \cos(c_1 + \frac{pt}{2}) + b]}. \quad (3.22)$$

Here b is the same as in equation (3.14) for the case $I > 0$ and the integration constants in terms of initial values of the ESR are,

$$c_1 = \tan^{-1} \left(\frac{2\theta_0}{p - b\sqrt{-I_0}} \right), \quad (3.23)$$

$$\{c_2, c_3, c_4\} = \{\sigma_{+0}, \sigma_{\times 0}, \omega_0\} (a \cos c_1 + b). \quad (3.24)$$

All the solutions mentioned above become singular in finite time. Note, however, that this

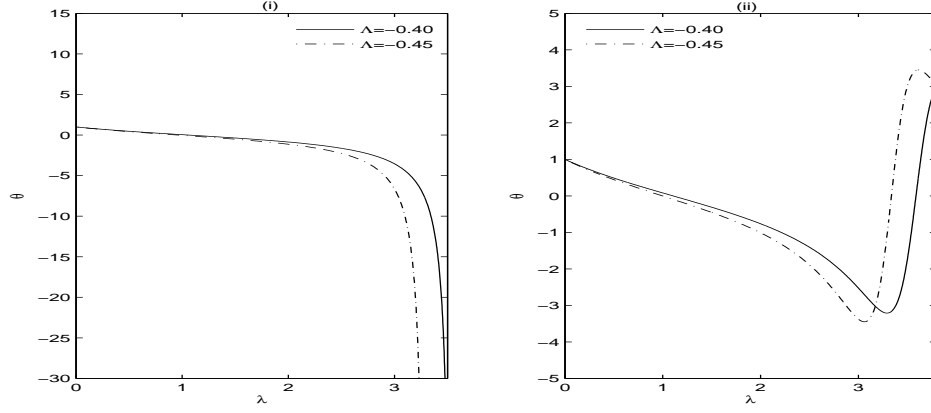


FIG. 1: Effect of Λ on the evolution of the expansion scalar $\theta(\lambda)$ for the cases (i) without initial rotation, and (ii) with initial rotation ($\omega_0 = 0.2$).

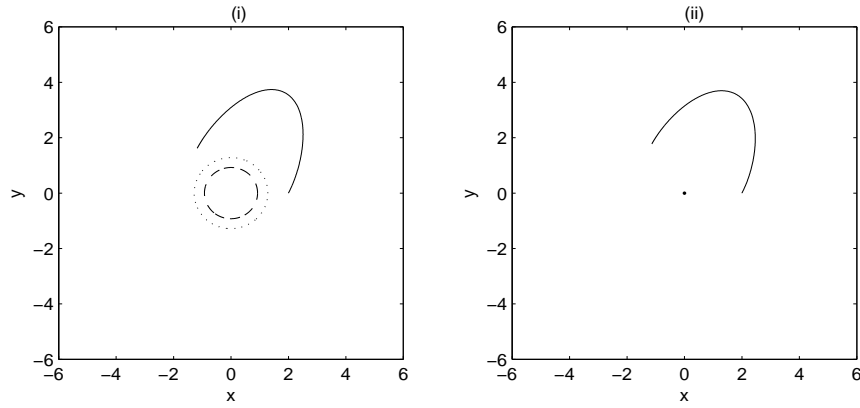


FIG. 2: Orbits around the BTZ black hole without initial rotation for (i) with horizon ($\Lambda = -0.4$), and (ii) without horizon ($\Lambda = -0.45$). The dotted and dashed lines indicate, respectively, the outer (r_+) and inner (r_-) horizons in the left panel above.

is a *singularity of the congruence*. One may note that the kinematics of geodesic flows, as governed by the Raychaudhuri equations, in the freely falling frame, is independent of the

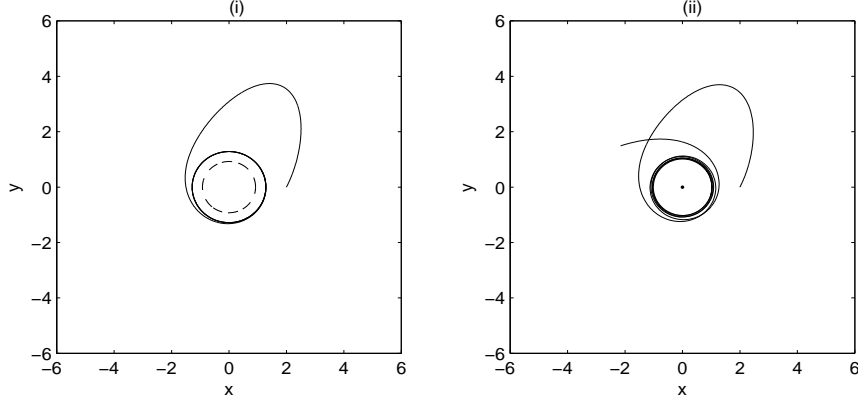


FIG. 3: Orbits around the BTZ black hole with initial rotation ($\omega_0 = 0.2$) for (i) with horizon ($\Lambda = -0.4$), and (ii) without horizon ($\Lambda = -0.45$).

value of the parameters M , J , E and L . However, the geodesics are dependent on these parameters and we use the following values of parameters $M = 1$, $E = 2$, $L = 2$ and $J = 1.5$ for visualising the orbits around the BTZ black hole. For the kinematics, we consider two cases with $\Lambda = -0.40$ and $\Lambda = -0.45$ which correspond to the BTZ black hole with and without horizon, respectively. For these two cases, we observe the effect of rotation on the expansion scalar θ in Fig.1. The corresponding orbits are presented in Fig.2 (without initial rotation) and Fig.3 (with initial rotation). It is observed that the geodesic congruence is necessarily singular without initial rotation (see Fig.1), a fact which can also be inferred from the analytical solutions. Geodesic focusing always occurs earlier with the shrinking of the horizon of the BTZ black hole, due to an increase in the value of $|\Lambda|$, as shown in Fig.1(i). However beyond a critical value of the initial rotation, the congruence singularity is removed irrespective of the presence/absence of the horizon, as shown in Fig.1(ii). It may be noted that the Fig.1(i) corresponds to the exact solutions of the ESR variables represented by equations (3.12) and (3.13) for the case $I > 0$. Further, Fig.1(ii) corresponds to the exact solutions of the ESR variables represented by equations (3.21) and (3.22) for the case $I < 0$.

It might seem that the expression for θ in (3.11) does have a dependence on the parameters M , J , E , L among others. However, one can show using the solutions of the geodesic equations (i.e., $r(\lambda)$) in (3.11) that the final result coincides with one of the solutions obtained above in this section. Thus, it is surprising to note that, while the geodesic trajectories depend on the parameters M , J , etc., the ESR variables depend only on $|\Lambda|$. This is essentially an effect of the spacetime dimension ($2+1$ here) where the Weyl tensor is

identically zero and no explicit dependence on M , J appear in the ESR evolution equations.

IV. THE KERR BLACK HOLE SPACETIME

In this section, we quickly recall the well-known Kerr black hole spacetime (in the Boyer–Lindquist coordinates, which are analogous to the Schwarzschild coordinates for a non-rotating black hole) with mass M and angular momentum J as characterized by the following stationary and axisymmetric metric,

$$ds^2 = - \left(1 - \frac{2Mr}{\rho^2} \right) dt^2 - \frac{4Mar \sin^2 \psi}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\psi^2 + \left(r^2 + a^2 + \frac{2Mar a^2 \sin^2 \psi}{\rho^2} \right) \sin^2 \psi d\phi^2, \quad (4.1)$$

Here $a = J/M$, $\rho^2 = r^2 + a^2 \cos^2 \psi$ and $\Delta = r^2 - 2Mr + a^2$. With $a = 0$, the Kerr metric (4.1) reduces to the Schwarzschild. The geodesic equations corresponding to the metric (4.1) are well-known (we do not repeat them here). However, analytical expressions of first integrals of the geodesic equations are not known for the general case (except for the choice of $\psi = \pi/2$ [14]). In our work here, we are interested in timelike geodesic congruences for which the constraint $u^\alpha u_\alpha = -1$ is satisfied, i.e.,

$$g_{00}\dot{t}^2 + g_{11}\dot{r}^2 + g_{22}\dot{\psi}^2 + g_{33}\dot{\phi}^2 + 2g_{03}\dot{t}\dot{\phi} + 1 = 0. \quad (4.2)$$

Once the initial conditions on the velocity vector components are specified in such a way that this constraint is satisfied, it will be satisfied throughout the geodesic evolution.

V. ESR EVOLUTION

A. The Raychaudhuri equations

Consider a congruence of geodesics with the associated time like vector field u^α , tangent to the geodesic at each point and satisfying equation (4.2). A transverse metric $h_{\alpha\beta}$ on a spacelike hypersurface can be written by decomposing the spacetime metric ($g_{\alpha\beta}$) as defined through equation (4.1) into the longitudinal ($-u_\alpha u_\beta$) and transverse part (i.e., $h_{\alpha\beta}$),

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta; \quad (\alpha, \beta = 0, 1, 2, 3). \quad (5.1)$$

The transverse metric satisfies $u^\alpha h_{\alpha\beta} = 0$ which means that $h_{\alpha\beta}$ is orthogonal to u^α and the transverse spacelike hypersurface represents the local rest frame of a freely falling observer in the given spacetime. The evolution of the ESR on such a transverse hypersurface, can be investigated by using a tensor $B_{\alpha\beta}$, which can be decomposed into its trace, symmetric traceless and antisymmetric parts as follows,

$$B_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \quad (5.2)$$

where $\theta = B^\alpha_\alpha$ is the expansion scalar, $\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \theta h_{\alpha\beta}/3$ the shear tensor, and $\omega_{\alpha\beta} = B_{[\alpha\beta]}$, the rotation tensor with the brackets $()$ and $[\]$ denoting symmetrisation and antisymmetrisation, respectively. The shear and rotation tensors also satisfy $h^{\alpha\beta} \sigma_{\alpha\beta} = 0$ and $h^{\alpha\beta} \omega_{\alpha\beta} = 0$. The evolution equation for $B_{\alpha\beta}$ takes the following form,

$$\dot{B}_{\alpha\beta} + B_{\alpha\gamma} B^\gamma_\beta = -R_{\alpha\eta\beta\delta} u^\eta u^\delta, \quad (5.3)$$

Using the equation (5.3) and following standard methods [1, 7, 9], the Raychaudhuri equations for the ESR variables in the Kerr black hole spacetime, given by equation (4.1) turn out to be,

$$\dot{\theta} + \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 = 0, \quad (5.4)$$

$$\dot{\sigma}_{\alpha\beta} + \frac{2}{3}\theta \sigma_{\alpha\beta} + \sigma_{\alpha\gamma} \sigma^\gamma_\beta + \omega_{\alpha\gamma} \omega^\gamma_\beta + \frac{1}{3}(\sigma^2 - \omega^2) h_{\alpha\beta} + C_{\alpha\eta\beta\delta} u^\eta u^\delta = 0, \quad (5.5)$$

$$\dot{\omega}_{\alpha\beta} + \frac{2}{3}\theta \omega_{\alpha\beta} + \sigma_\alpha^\gamma \omega_{\gamma\beta} + \omega_\alpha^\gamma \sigma_{\gamma\beta} = 0, \quad (5.6)$$

where $\sigma^2 = \sigma_{\alpha\beta} \sigma^{\alpha\beta}$, $\omega^2 = \omega_{\alpha\beta} \omega^{\alpha\beta}$ and $C_{\alpha\beta\eta\delta} \equiv R_{\alpha\eta\beta\delta}$ is the Weyl tensor for the present case. One can notice the absence of the geometric part (i.e., $R_{\alpha\beta} u^\alpha u^\beta$) in equation (5.4) since the Kerr spacetime is Ricci flat. Further, the nonzero Weyl in equation (5.5), with its dependence on the rotation parameter a (and also M) will lead to a dependence on a in the evolution of the ESR.

B. Visualisation of ESR evolution

In the absence of analytical solutions for the ESR variables in this case, we have directly solved (5.3) numerically. In order to represent these kinematical quantities at any point, we consider a freely falling (Fermi) normal frame with the basis vectors E_μ^α , $\mu = 0, \dots, 3$ (with $E_0^\alpha = \hat{u}^\alpha$) which are parallelly transported. To construct such frames numerically, we solve

the differential equations $u^\beta \nabla_\beta E_\mu^\alpha = 0$ (with initial conditions of an orthonormal frame) simultaneously with (5.3) [7]. The tensor $B_{\alpha\beta}$ in the Fermi basis may be represented as,

$$B_{\alpha\beta} = \left(\frac{1}{3}\theta + \sigma_{11}\right)e_\alpha^1 e_\beta^1 + \left(\frac{1}{3}\theta + \sigma_{22}\right)e_\alpha^2 e_\beta^2 + \left(\frac{1}{3}\theta - \sigma_{11} - \sigma_{22}\right)e_\alpha^3 e_\beta^3 + (\sigma_{12} - \omega_3)e_\alpha^1 e_\beta^2 + (\sigma_{21} + \omega_3)e_\alpha^2 e_\beta^1 + (\sigma_{13} + \omega_2)e_\alpha^1 e_\beta^3 + (\sigma_{31} - \omega_2)e_\alpha^3 e_\beta^1 + (\sigma_{23} - \omega_1)e_\alpha^2 e_\beta^3 + (\sigma_{32} + \omega_1)e_\alpha^3 e_\beta^2, \quad (5.7)$$

where e_α^μ are co-frame basis which satisfy the relation $e_\alpha^\mu E_\nu^\alpha = \delta_\nu^\mu$. The ESR variables can now be constructed from the evolution tensor (5.7), using the basis vectors, as,

$$\theta = B_{\alpha\beta} h^{\alpha\beta} \equiv B_{\alpha\beta} g^{\alpha\beta}, \quad (5.8)$$

$$\sigma_{11} = B_{\alpha\beta} E_1^\alpha E_1^\beta - \frac{1}{3}\theta, \quad (5.9)$$

$$\sigma_{22} = B_{\alpha\beta} E_2^\alpha E_2^\beta - \frac{1}{3}\theta, \quad (5.10)$$

$$\sigma_{12} = \frac{1}{2}(B_{\alpha\beta} E_1^\alpha E_2^\beta + B_{\alpha\beta} E_2^\alpha E_1^\beta), \quad (5.11)$$

$$\sigma_{13} = \frac{1}{2}(B_{\alpha\beta} E_1^\alpha E_3^\beta + B_{\alpha\beta} E_3^\alpha E_1^\beta), \quad (5.12)$$

$$\sigma_{23} = \frac{1}{2}(B_{\alpha\beta} E_2^\alpha E_3^\beta + B_{\alpha\beta} E_3^\alpha E_2^\beta), \quad (5.13)$$

$$\omega_1 = \frac{1}{2}(B_{\alpha\beta} E_3^\alpha E_2^\beta - B_{\alpha\beta} E_2^\alpha E_3^\beta), \quad (5.14)$$

$$\omega_2 = \frac{1}{2}(B_{\alpha\beta} E_1^\alpha E_3^\beta - B_{\alpha\beta} E_3^\alpha E_1^\beta), \quad (5.15)$$

$$\omega_3 = \frac{1}{2}(B_{\alpha\beta} E_2^\alpha E_1^\beta - B_{\alpha\beta} E_1^\alpha E_2^\beta). \quad (5.16)$$

In order to understand the focusing/defocusing behaviour of a timelike geodesic congruence, let us redefine the expansion scalar as $\theta = 3\dot{F}/F$ where the dot indicates the derivative with respect to the affine parameter λ . The equation (5.4) can now be written in the following Hill-type equation,

$$\ddot{F} + H F = 0, \quad (5.17)$$

where $H = (\sigma^2 - \omega^2)/3$ with $\sigma^2 = 2(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 + \sigma_{11}\sigma_{22})$ and $\omega^2 = 2(\omega_1^2 + \omega_2^2 + \omega_3^2)$. If $F \rightarrow 0$ in finite time, we have a finite time singularity in θ with focusing (defocusing) if $\dot{F} < 0$ ($\dot{F} > 0$). The signature of the invariant quantity H is useful in understanding the individual roles of shear and rotation in the occurrence of geodesic focusing/defocusing. When H is positive definite (i.e., $\sigma^2 > \omega^2$), there exists conjugate points and geodesic

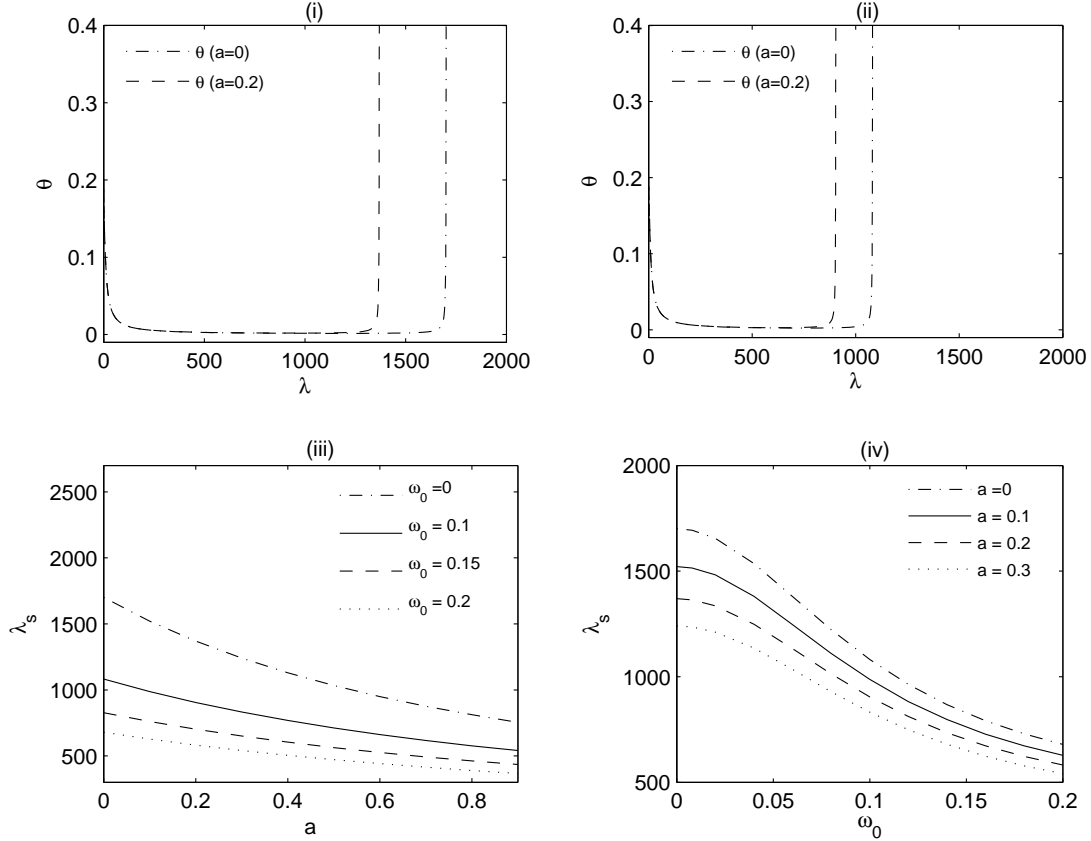


FIG. 4: Defocusing (for $\theta_0 > \theta_c$) for the cases (i) without initial rotation, and (ii) with initial rotation ($\omega_0 = 0.1$). The effect of (iii) angular momentum, and (iv) initial rotation on time of approach to singularity (λ_s). The initial shear in all cases is zero.

focusing/defocusing takes place. On the other hand, no finite time singularity exists for an initially non-contracting congruence (i.e., $\theta_0 \geq 0$) if H is negative definite. For $\theta_0 < 0$, there exists a critical value below which focusing/defocusing will take place.

In the following, the evolution of the ESR variables in the Kerr black hole background, under different conditions, are presented and compared with those in the Schwarzschild background for some interesting cases.

We consider the case when $\psi = \pi/2$ (equatorial section). We have observed that for an initially expanding congruence, geodesic defocusing takes place (i.e., $\theta \rightarrow \infty$) and there exists a critical value of the expansion scalar θ_c below which there is a focusing (i.e., $\theta \rightarrow -\infty$). It is important to mention that with the change in the initial conditions on shear and rotation, the critical value of the initial expansion scalar will also change. The results with no initial shear and rotation are presented in Figs. 4(i) and (ii) for θ_0 above the

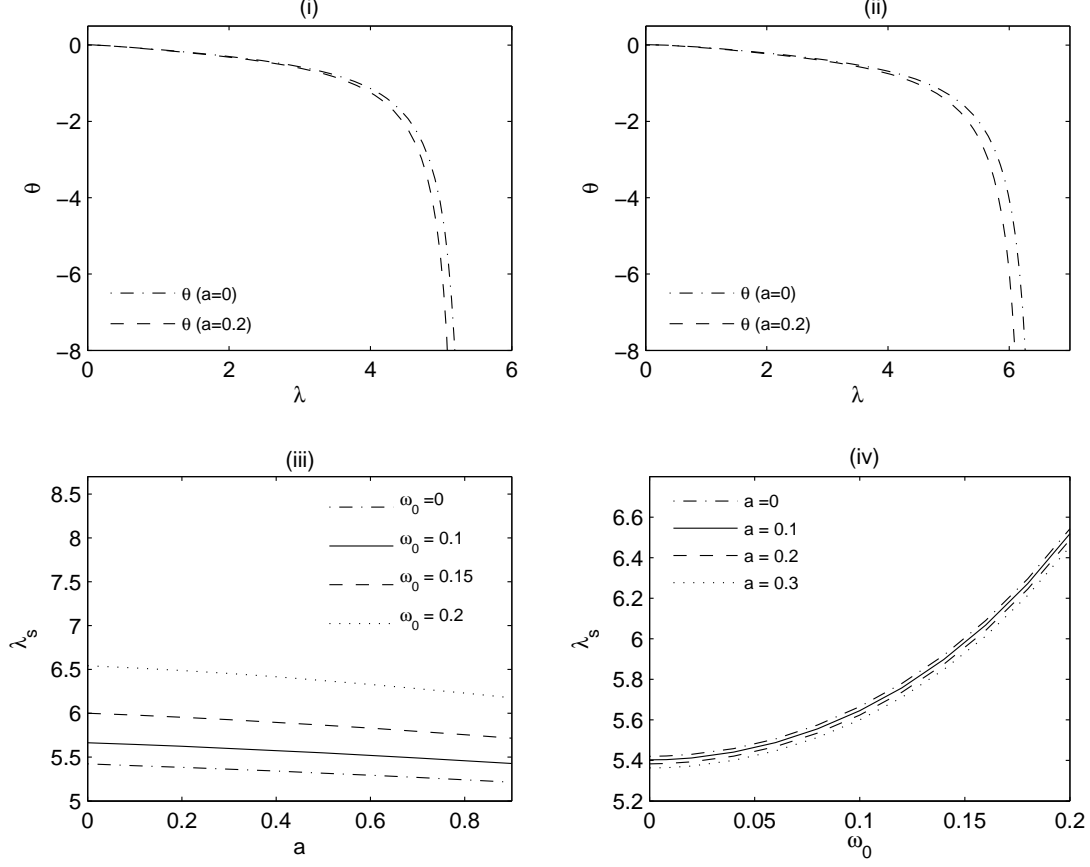


FIG. 5: Focusing (for $\theta_0 < \theta_c$) for the cases (i) without initial rotation, and (ii) with initial rotation ($\omega_0 = 0.2$). The effect of (iii) angular momentum, and (iv) initial rotation on time of approach to singularity (λ_s). The initial shear is $\sigma_0 = 0.28$.

critical initial expansion (defocusing), and Figs. 5(i) and (ii) for θ_0 below the critical initial expansion (focusing) for both Kerr and Schwarzschild ($a = 0$) black holes. In order to see the effects of the angular momentum a and the initial rotation ω_0 of the congruence, we have calculated the time to singularity λ_s for the super-critical and sub-critical values of θ_0 and presented them, respectively, in Figs. 4(iii) and (iv), and Figs. 5(iii) and (iv). For the case in Fig. 4, we observe that both angular momentum and initial rotation of the congruence assist defocusing. On the other hand in Fig. 5, it is observed that with increase in ω_0 , the congruence focuses at a later instant, as expected, while an opposite trend is observed for increase in a . Thus, we conclude that the angular momentum a advances the time to singularity (irrespective of its type), while the initial rotation ω_0 of the congruence advances defocusing and delays focusing. For this reason, for the same set of initial conditions,

focusing/defocusing occurs earlier in the Kerr black hole as compared to the Schwarzschild. We have also noticed that the effect of initial shear on geodesic focusing and defocusing for the Kerr case is the same as that for the Schwarzschild (results are not presented here). Focusing occurs at an earlier instant of time when we have some initial shear in comparison to the case with zero initial shear.

Since all values of M and a do not correspond to a Kerr black hole and the horizon exists only when $a \leq M$, the limiting value of the angular momentum is $a = M$ (extreme Kerr). We have observed that for the extreme Kerr, all the features of the geodesic focusing and defocusing remain qualitatively similar as those for the non-extreme Kerr case, though with a change in the scale.

VI. SUMMARY AND CONCLUSIONS

In this article, we have investigated the kinematics of of timelike geodesic congruences in the background of the BTZ and Kerr black hole spacetimes. The important conclusions are summarised below:

- Apart from the cosmological constant (Λ), the other parameters of the BTZ black hole spacetime, do not have any direct role in the kinematics of the geodesic congruence. However, the geodesic trajectories surely depend on the black hole parameters.
- An increase in Λ in the BTZ case, which implies a larger negative curvature, enhances focusing of the congruence.
- The notion of focusing remains qualitatively similar in the BTZ black hole background for the cases with or without horizons.
- The angular momentum parameter a of the Kerr black hole assists both focusing and defocusing when the respective cases occur. Thus, physically, the larger the spin of the black hole, the earlier the focusing/defocusing of the geodesic congruence occurs. This effect is absent in the BTZ black hole case.
- In contrast with the role of a , an increase in the initial rotation, ω_0 , of the congruence assists defocusing but delays focusing in both the $(2 + 1)$ and $(3 + 1)$ dimensional rotating black holes. This observation is similar to results obtained previously [5]-[7].

The tidal distortion due to the presence of mass nearby, or accretion of matter into a rotating black hole deserves careful attention. The kinematics of inflows can be conveniently studied using the Raychaudhuri equations in these backgrounds. The effect of the black hole angular momentum parameter on the focussing/defocussing of geodesic congruences, as has been observed in this work, can have an interesting role in the accretion process and may produce observable effects. Further, for a more comprehensive understanding, the kinematics of flows in maximally extended coordinate systems, for black hole spacetimes, would be interesting to investigate. Finally, the study of kinematics of null geodesic flows in such spacetimes is surely worth trying out in future.

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